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THE STRUCTURE AND MEASURE OF SINGULAR SETS OF SOLUTIONS TO ELLIPTIC EQUATIONS

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For a harmonic function in an open set in $\mathbb{R}^2$, the subset of critical points in the nodal set is exactly the singular part of the nodal set. For this reason, this subset of critical points is called the singular set. It is well known that the singular set of a 2-dimensional harmonic function is isolated. Around each point in the singular set, the nodal set consists of finitely many analytic curves intersecting at this point, forming equal angles. In fact, the number of singular points can be estimated in terms of the growth of the harmonic function. One way to do this is to identify $\mathbb{R}^2$ as $\mathbb{C}$ and then the singular set can be identified as the zero set of some holomorphic function.

In this note, we shall study the critical nodal sets, or the singular sets, of solutions to homogeneous elliptic equations of the second order. To be specific, we shall study the structure and the size of the critical nodal sets. Throughout the paper, we shall assume that u is at least a nonzero C^2 solution in $B_1 \subset \mathbb{R}^n$ to the following elliptic equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u = 0, \tag{0.1}$$

where the coefficients satisfy the following assumptions

$$\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^n,\ x \in B_1,\\ &\sum_{i,j=1}^{n} |a_{ij}(x)| + \sum_{i=1}^{n} |b_i(x)| + |c(x)| \leq \kappa, \quad \text{for any } x \in B_1, \end{aligned} \tag{0.2}$$

and

$$\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| \leq K|x-y|, \quad \text{for any } x, y \in B_1, \tag{0.3}$$

for some positive constants λ, κ and K. The Lipschitz condition (0.3) for the leading coefficients is essential. It implies the unique continuation for the operator $\mathcal{L}$. In other words, if a solution u to (0.1) vanishes to an infinite order at a point in B_1, then u is identically zero. For details, see [7].

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Now we define the nodal set and the singular set by

$$\mathcal{N}(u) = \{p \in B_1; u(p) = 0\},$$
$$\mathcal{S}(u) = \{p \in B_1; u(p) = |\partial u(p)| = 0\}.$$

By the implicit function theorem, $\mathcal{N}(u) \setminus \mathcal{S}(u)$ is an $(n-1)$-dimensional hypersurface, at least locally. In this note, we shall study $\mathcal{S}(u)$. We shall prove that $\mathcal{S}(u)$ is $(n-2)$-dimensional and its $(n-2)$-dimensional measure is bounded in terms of the frequency.

1. The Structure of Singular Sets

We first begin with a simple case.

Lemma 1.1. *Let a_{ij}, b_i, c be smooth in $B_1 \subset \mathbb{R}^n$ and u be a smooth solution to (0.1) in B_1. Then $\mathcal{S}(u)$ is contained in a countable union of $(n-2)$-dimensional smooth manifolds.*

Proof. For any $p \in B_1$, we set the vanishing order $\mathcal{O}(p)$ of u at p as

$$\mathcal{O}(p) = \mathcal{O}_u(p) = \{d;\ \partial^\nu u(p) = 0 \text{ for any } |\nu| < d,\ \partial^{\nu_0} u(p) \neq 0 \text{ for some } |\nu_0| = d\}.$$

Obviously, $\mathcal{O}(p) \geq 2$ for $p \in \mathcal{S}(u)$. For any $d \geq 2$, we set

$$\mathcal{S}_d(u) = \{p \in B_1;\ \mathcal{O}(p) = d\}.$$

Then we have

$$\mathcal{S}(u) = \bigcup_{d\geq 2} \mathcal{S}_d(u). \tag{1.1}$$

This is a finite union by the unique continuation. We shall prove that each $\mathcal{S}_d(u)$ is $(n-2)$-dimensional for each fixed $d \geq 2$.

For any $p \in \mathcal{S}_d(u)$, there exists a $|\beta| = d-2$ such that $\partial^2 v(p) \neq 0$ for $v = \partial^\beta u$. Now applying ∂^β to (0.1) and evaluating at p, we obtain

$$\sum_{i,j=1}^n a_{ij}(p)\partial_{ij} v(p) = 0.$$

First, the Hessian matrix $(\partial^2 v(p))$ has a nonzero eigenvalue. Next, we may diagonalize

$$(\partial^2 v(p)) = \operatorname{diag}(\lambda_1, \cdots, \lambda_n).$$

Then we have

$$a_1(p)\lambda_1 + \cdots + a_n(p)\lambda_n = 0,$$

for some positive constants $a_1(p), \cdots, a_n(p)$. By assuming $\lambda_1 \neq 0$, we have another nonzero eigenvalue and hence we may assume $\lambda_2 \neq 0$. Note

$$\partial\partial_1 v(p) = (\lambda_1, 0 \cdots, 0), \quad \partial\partial_2 v(p) = (0, \lambda_2, 0, \cdots, 0).$$

By applying the implicit function theorem to $\partial_1 v$ and $\partial_2 v$, we conclude that $\{\partial_1 v = 0, \partial_2 v = 0\}$ is an $(n-2)$-dimensional manifold in a neighborhood of p. Obviously, this manifold contains $S_d(u)$ in a neighborhood of p. This finishes the proof. $\square$

Now, we shall discuss nonsmooth solutions. First, we shall generalize the notion of the vanishing order. Suppose u is a solution to (0.1). By the unique continuation, for any $p \in B_1$ there exists an integer d such that

$$\limsup_{x\to p} \frac{|u(x)|}{|x-p|^d} < \infty,$$
$$\limsup_{x\to p} \frac{|u(x)|}{|x-p|^{d+1}} = \infty.$$

Bers [1] proved that there exists a nonzero homogeneous polynomial P of degree d such that

$$u(x) = P(x-p) + o(|x-p|^d).$$

Naturally the integer d, the degree of the polynomial P, is called *the vanishing order* of u at p, denoted by $\mathcal{O}(p)$ or $\mathcal{O}_u(p)$. For convenience we call the nonzero homogeneous polynomial *the leading polynomial* of u at p. We have following results concerning the vanishing order and the leading polynomial.

Lemma 1.2. *Let u be a C^2 solution of (0.1) with (0.2) and (0.3) and P be the leading polynomial of u at 0, with $d = degP$. Then there hold for any $\alpha \in (0,1)$*

$$\sum_{i,j=0}^{n} a_{ij}(0)\partial_{ij}P = 0 \quad \text{in } \mathbb{R}^n,$$
$$|P(x)| \le C\|u\|_{L^2(B_1)}|x|^d \quad \text{in } B_1,$$
$$|u(x) - P(x)| \le C\|u\|_{L^2(B_1)}|x|^{d+\alpha} \quad \text{in } B_{\frac{1}{2}}(0),$$

and

$$\sum_{i=1}^{2} r^i\|D^i(u-P)\|_{L^2(B_r)} \le C\|u\|_{L^2(B_1)}r^{d+\alpha+\frac{n}{2}} \quad \text{for any } r \le \frac{1}{2},$$

where C is a constant depending only on $n, d, \lambda, \alpha, \kappa$ and K.

Lemma 1.3. *Suppose that $\{\mathcal{L}_k\}_{k=0}^{\infty}$ is a family of elliptic operators in B_1 of the form (0.1) satisfying (0.2) and (0.3) and that u_k is a C^2 solution of $\mathcal{L}_k u_k = 0$ in B_1 for $k = 0, 1, 2, \cdots$. Suppose that $\mathcal{L}_k \to \mathcal{L}_0$ in the sense that the corresponding coefficients converge uniformly and that $u_k \to u_0$ in $L^\infty(B_1)$. Then there holds*

$$\limsup_{k\to\infty} \mathcal{O}_{u_k}(0) \le \mathcal{O}_{u_0}(0). \tag{1.2}$$

If, in addition, $\mathcal{O}_{u_k}(0) = d$ and P_k is the leading polynomial of u_k at 0 for $k = 1, 2, \cdots$, then the following conclusions hold:

(i) if $\mathcal{O}_{u_0}(0) > d$, then

$$P_k \to 0 \quad \text{uniformly in } B_1 \text{ as } k \to \infty;$$

(ii) if $\mathcal{O}_{u_0}(0) = d$, then

$$P_k \to P_0 \quad \text{uniformly in } B_1(0) \text{ as } k \to \infty,$$

where P_0 is the leading polynomial of u_0 at 0.

The proof is quite complicated. In [9], we first proved Lemma 1.2 and Lemma 1.3 by using the monotonicity of the frequency function [7]. Such a method is limited to elliptic equations of the second order. Later on, we proved Lemma 1.2 and Lemma 1.3 by using the singular integrals. In fact, we proved these results for elliptic equations of the arbitrary order. For details, see [10].

Now we state the main result in this section. It is taken from [9].

Theorem 1.4. *Let u be a C^2 solution to (0.1) with (0.2) and (0.3). Then there exists the following decomposition*

$$\mathcal{S}(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^j(u),$$

where each $\mathcal{S}^j(u)$ is on a countable union of j-dimensional C^1 graphs, $j = 0, 1, \cdots, n-2$.

Proof. The proof consists of several steps. For each fixed $d \geq 2$, we shall study

$$\mathcal{S}_d(u) = \{p \in \mathcal{S}(u); \mathcal{O}(u) = d\}.$$

Step 1. We use Lemma 1.2 to study the local behavior at each point.

For each point $y \in B_{\frac{1}{2}} \cap \mathcal{S}_d(u)$, set for any $r \in (0, \frac{1-|y|}{2})$,

$$u_{y,r}(x) = \frac{u(y+rx)}{(\fint_{\partial B_r(y)} |u|^2)^{\frac{1}{2}}} \quad \text{for any } x \in B_2. \tag{1.3}$$

Then by Lemma 1.2, we have

$$u_{y,r} \to P \quad \text{in } L^2(B_2) \quad \text{as } r \to 0, \tag{1.4}$$

where $P = P_y$ is a d-degree non-zero homogeneous polynomial satisfying

$$\sum_{i,j=1}^{n} a_{ij}(0)\partial_{ij} P = 0. \tag{1.5}$$

Moreover, $\|P\|_{L^2(\partial B_1)} = 1$. Note P is the normalized leading polynomial of u at y.

Since P is a d-degree non-zero homogeneous polynomial, we have

$$\mathcal{S}_d(P) = \{x; \partial^\nu P(x) = 0, \text{ for any } |\nu| \leq d-1\}.$$

Obviously $0 \in \mathcal{S}_d(P)$ by the homogeneity of P. It is easy to see that $\mathcal{S}_d(P)$ is a linear subspace and

$$(1.6) \qquad P(x) = P(x+z) \quad \text{for any } x \in \mathbb{R}^n \text{ and } z \in \mathcal{S}_d(P).$$

Next, we observe that $\dim\mathcal{S}_d(P) \le n-2$ for $d \ge 2$. In fact, (1.6) implies P is a function of n-$\dim\mathcal{S}_d(P)$ variables. If $\dim\mathcal{S}_d(P) = n-1$, P would be a d-degree monomial of one variable satisfying the equation (1.5). Hence $d < 2$.

Step 2. We define for each $j = 0, 1, 2, \dots, n-2$,

$$\mathcal{S}_d^j(u) = \{y \in \mathcal{S}_d(u);\ \dim\mathcal{S}_d(P_y) = j\}.$$

We claim that $\mathcal{S}_d^j(u)$ is on a countable union of j-dimensional C^1 graphs. In fact, we shall prove that for any $y \in \mathcal{S}_d^j(u)$ there exists an $r = r(y)$ such that $\mathcal{S}_d^j(u) \cap B_r(y)$ is contained in a (single piece of) j-dimensional C^1 graph.

To show this, we let ℓ_y be the j-dimensional linear subspace $\mathcal{S}_d(P_y)$ for any $y \in \mathcal{S}_d^j(u)$. For any $\{y_k\} \subset \mathcal{S}_d^j(u)$ with $y_k \to y$, we first prove

$$(1.7) \qquad \text{Angle } < \overline{yy_k},\ \ell_y > \to 0.$$

To prove (1.7), we may assume $y = 0$ and $p_k = \frac{y_k}{|y_k|} \to \xi \in \mathbb{S}^{n-1}$. Note $p_k \in \mathcal{S}_d(u_{0,|y_k|})$ for

$$u_{0,|y_k|}(x) = \frac{u(|y_k|x)}{\left(\fint_{\partial B_{|y_k|}(0)} u^2\right)^{\frac{1}{2}}}.$$

See (1.3) for notations. We may show by an elementary calculation that

$$\mathcal{L}_k u_{0,|y_k|} = 0,$$

where $\mathcal{L}_k$ is some second order elliptic operator with a similar structure as $\mathcal{L}$. Moreover, for $\mathcal{L}$ as in (0.1), we have

$$\mathcal{L}_k \to \mathcal{L}_0 \equiv \sum_{i,j=1}^n a_{ij}(0)\partial_{ij},$$

in the sense that corresponding coefficients converge uniformly. Then by applying Lemma 1.3, we obtain that P_y vanishes at ξ with an order at least d, i.e.,

$$\mathcal{O}_{P_y}(\xi) \ge d.$$

Since P_y is a d-degree homogeneous polynomial, then $\mathcal{O}_{P_y}(\xi) = d$ and $\xi \in \ell_y$. This implies (1.7).

By (1.7), we obtain that for any $y \in \mathcal{S}_d^j(u)$ and small $\varepsilon > 0$ there exists an $r = r(y, \varepsilon)$ such that

$$(1.8) \qquad \mathcal{S}_d^j(u) \cap B_r(y) \subset B_r(y) \cap C_\varepsilon(\ell_y),$$

where

$$C_\varepsilon(\ell_y) = \{z \in \mathbb{R}^n; \operatorname{dist}(z, \ell_y) \le \varepsilon|z|\}.$$

Let P_k and P be leading polynomials of u at y_k and $y = 0$, respectively. By Lemma 1.3, we have

$$P_k \to P \quad \text{uniformly in } C^d(B_1).$$

This implies

$$\ell_{y_k} \to \ell_y \quad \text{as } k \to \infty,$$

as subspaces in $\mathbb{R}^n$. By an argument similar as proving (1.7), we may prove that the constant r in (1.8) can be chosen uniformly for any point $z \in \mathcal{L}_d^j(u)$ in a neighborhood of y. In other words, for any $y \in \mathcal{S}_d^j(u)$ and any small $\varepsilon > 0$ there exists an $r = r(\varepsilon, y)$ such that

$$\mathcal{S}_d^j(u) \cap B_r(z) \subset B_r(z) \cap C_\varepsilon(\ell_z) \quad \text{for any } z \in \mathcal{S}_d^j(u) \cap B_r(y).$$

For $\varepsilon > 0$ small enough, this clearly implies that $\mathcal{S}_d^j(u) \cap B_r(y)$ is contained in a j-dimensional Lipschitz graph. By (1.7) this graph is C^1. □

Remark 1.5. In fact, we can prove $\mathcal{S}^{n-2}(u)$ is on a countable union of $(n-2)$-dimensional $C^{1,\beta}$ manifolds, for some $0 < \beta < 1$.

Now we write a corollary of Theorem 1.4.

Corollary 1.6. *Let u be a solution as in Theorem 1.4. Then there holds*

$$\mathcal{S}(u) = \mathcal{S}_*(u) \bigcap \mathcal{S}^*(u),$$

where the Hausdorff dimension of $\mathcal{S}^*(u)$ is at most $n-3$, and for any $p \in \mathcal{S}_*(u)$ the leading polynomial of u at p is a polynomial of two variables after some rotation of coordinates.

To conclude the present section, we illustrate by an example that in $\mathbb{R}^3$ the singular set can be any closed subset in a line segment.

2. The Measure of Singular Sets

In this section, we shall discuss the geometric measure of singular sets.

We begin with a simple example. Consider a homogenous harmonic polynomial of degree d in $\mathbb{R}^2$. By using the polar coordinate $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ in $\mathbb{R}^2 = \{(x_1, x_2)\}$, we may assume $P(x) = r^d \cos d\theta$. A direct calculation shows that

$$\partial_1 P = dr^{d-1}\cos(d-1)\theta, \ \partial_2 P = -dr^{d-1}\sin(d-1)\theta.$$

Therefore both $\partial_1 P$ and $\partial_2 P$ are products of $d-1$ different homogeneous linear functions. Now assume u is a smooth perturbation of P in B_1. Then it is not hard to imagine that the critical set of u has at most $(d-1)^2$ points in B_1. As we shall see, this is quite difficult to prove.

This simple observation illustrates that the size of singular sets of harmonic polynomials depend on the degree. In order to obtain a measure estimate of singular sets of solutions to general elliptic equations, we first need to introduce a quantity to measure the growth of solutions.

Suppose that $\mathcal{L}$ is an elliptic operator of the form (0.1) satisfying (0.2) and (0.3) and that u is a C^2 solution of $\mathcal{L}u = 0$ in B_1. Set

$$N = \frac{\int_{B_1} |\partial u|^2}{\int_{\partial B_1 u^2}}. \tag{2.1}$$

It is proved in [7] that u satisfies

$$⨍_{B_{2r}(p)} u^2(x)dx \leq 4^{c_0 N} ⨍_{B_r(x_0)} u^2(x)\,dx, \quad \text{for any } x_0 \in B_{\frac{1}{2}},\ r < r_0,$$

where c_0 and $r_0 < 1/3$ are positive constants depending only on λ, κ, K and n. Here, we denote

$$⨍_{B_\rho(x_0)} u^2(x)dx \equiv \rho^{-n} \int_{B_\rho(x_0)} u^2(x)dx.$$

We then conclude that the vanishing order of u at any point $p \in B_{1/2}$ does not exceed $c_0 N$.

The quantity N in (2.1) is called the frequency of u in B_1. It controls the vanishing order of u. If u is a homogeneous harmonic polynomial, the frequency is exactly the degree. See [7] for a discussion of the frequency and related topics. In [16], Lin conjectured that

$$\mathcal{H}^{n-2}(\mathcal{S}(u) \cap B_{\frac{1}{2}}) \leq cN^2,$$

where c is a positive constant depending only on the elliptic operator $\mathcal{L}$.

The main result is the following theorem. It is taken from [12].

Theorem 2.1. *Suppose that $\mathcal{L}$ is an elliptic operator of the form (0.1) satisfying (0.2) and (0.3) and that u is a C^2 solution of $\mathcal{L}u = 0$ in B_1 with*

$$\frac{\int_{B_1} |\partial u|^2}{\int_{\partial B_1 u^2}} \leq N_0,$$

for some positive constant N_0. Then there exists a positive integer M, depending on N_0, λ, κ and K, such that if, in addition, $a_{ij}, b_i, c \in C^M(B_1)$, there holds

$$\mathcal{H}^{n-2}(\mathcal{S}(u) \cap B_{\frac{1}{2}}) \leq C,$$

where C is a positive constant depending on N_0, λ, κ, K and the C^M-norms of the coefficients a_{ij}, b_i and c.

The key result is the following lemma for functions in $\mathbb{R}^2$.

Lemma 2.2. *Let P be a homogeneous harmonic polynomial of degree $d \geq 2$ in $\mathbb{R}^2$. Then there exist positive constants δ and r, depending on P, such that for any $u \in C^{2d^2}(B_1)$ with*

$$|u - P|_{C^{2d^2}(B_1)} < \delta,$$

there holds

$$\#(|Du|^{-1}\{0\} \cap B_{\frac{1}{2}}) \leq c(d-1)^2,$$

where c is a universal constant.

The proof of Lemma 2.2 is based on the Weierstrass-Malgrange Preparation Theorem for finitely differentiable functions. See [12] for details.

Now we describe the proof of Theorem 2.1.

Proof of Theorem 2.1. The proof consists of several steps.

Step 1. Set

$$\mathcal{S}_*(u) = \{p \in \mathcal{S}(u);\ \text{the leading polynomial of } u \text{ at } p \text{ is}\\ \text{a polynomial of two variables by an appropriate rotation}\}.$$

By Corollary 1.6, we have

$$\mathcal{H}^{n-2}(\mathcal{S}(u) \setminus \mathcal{S}_*(u)) = 0.$$

Then for any $\varepsilon > 0$, there exist at most countably many balls $B_{r_i}(x_i)$ with $r_i \le \varepsilon$ and $x_i \in \mathcal{S}(u) \setminus \mathcal{S}_*(u)$ such that

$$\mathcal{S}(u) \setminus \mathcal{S}_*(u) \subset \bigcup_i B_{r_i}(x_i), \tag{2.2}$$

and

$$\sum r_i^{n-2} \le \gamma(\varepsilon, u), \tag{2.3}$$

where $\gamma(\varepsilon, u) \to 0$ as $\varepsilon \to 0$.

We claim for any $y \in \mathcal{S}_*(u) \cap B_{3/4}$, there exist $R = R(y,u)$, $r = r(y,u)$ and $c = c(y,u)$, with $r < R$, such

$$\mathcal{H}^{n-2}\{B_r(y) \cap \mathcal{S}(u)\} \le cr^{n-2}. \tag{2.4}$$

The proof of (2.4) is based on Lemma 2.2 and the fact that the degree of the leading polynomial at any $p \in \mathcal{S}_*(u)$ is at most $c_0 N$. We omit the details.

It is obvious that the collection of $\{B_{r_i}(x_i)\}$ and $\{B_{r(y)}(y)\}$, $y \in \mathcal{S}_*(u)$, covers $\mathcal{S}(u)$. By the compactness of $\mathcal{S}(u)$, there exist $x_i \in \mathcal{S}(u) \setminus \mathcal{S}_*(u)$, $i = 1, \cdots, k = k(\varepsilon, u)$, and $y_j \in \mathcal{S}_*(u)$, $j = 1, \cdots, l = l(\varepsilon, u)$, such that

$$\mathcal{S}(u) \cap B_{3/4} \subset \left(\cup_{i=1}^k B_{r_i}(x_i)\right) \bigcup \left(\cup_{j=1}^l B_{s_j}(y_j)\right), \tag{2.5}$$

with $r_i \le \varepsilon$, $i = 1, \cdots, k$, and $s_j \le \varepsilon$, $j = 1, \cdots, l$.

Step 2. In Step 1, The constant γ in (2.3) and c in (2.4) depend on u. To improve the results established in Step 1, we should work in a compact class of elliptic operators satisfying (0.1)-(0.3) and in a compact class of solutions with controlled frequency. Then by a compactness argument, we conclude the following result. Let u be as given in Theorem 2.1. For any $\varepsilon > 0$ there exist positive constants $C(\varepsilon)$ and $\gamma(\varepsilon)$, depending also on N_0, as well as λ, κ, K and n, with $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that there exists a collection of balls $\{B_{r_i}(x_i)\}$ with $r_i \le \varepsilon$ and $x_i \in \mathcal{S}(u)$ such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2} \setminus \cup B_{r_i}(x_i)\right) \le C(\varepsilon),$$

and
$$\sum r_i^{n-2} \leq \gamma(\varepsilon).$$
We emphasize that $C(\varepsilon)$ and $\gamma(\varepsilon)$ are independent of u.

Step 3. We use the standard iteration process to prove Theorem 2.1. To begin with, define
$$\phi_0 = \{B_{1/2}(0)\}\ .$$
Fix an $\varepsilon > 0$. We claim that we may find $\phi_1, \phi_2, \cdots$, each of which consists of a collection of balls, such that for any $\ell \geq 1$
$$\text{rad}(B) \leq \frac{(2\varepsilon)^\ell}{2} \quad \text{for any } B \in \phi_\ell,$$
$$\sum_{B\in\phi_\ell} [\text{rad}(B)]^{n-2} \leq \gamma(\varepsilon)^\ell,$$
and
$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap \bigcup_{B\in\phi_{\ell-1}} B \setminus \bigcup_{B\in\phi_\ell} B\right) \leq C(\varepsilon)[\gamma(\varepsilon)]^{\ell-1},$$
where $C(\varepsilon)$ and $\gamma(\varepsilon)$ are given in Step 2. Observe that
$$\begin{aligned}\mathcal{S}(u) \cap B_{1/2}(0) \subset &\bigcup_{\ell=1}^{\infty}\left(\mathcal{S}(u) \cap \Big(\bigcup_{B\in\phi_{\ell-1}} B \setminus \bigcup_{B\in\phi_\ell} B\Big)\right)\\ &\cup \bigcap_{\ell=0}^{\infty}\left(\mathcal{S}(u) \cap \bigcup_{j=\ell}^{\infty}\bigcup_{B\in\phi_j} B\right).\end{aligned}$$
Hence we have
$$\begin{aligned}\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_{1/2}(0)\right) &\leq C(\varepsilon)\left\{\sum_{\ell\geq1}[\gamma(\varepsilon)]^{\ell-1} + \inf_{\ell\geq1}\sum_{j=\ell}^{\infty}[\gamma(\varepsilon)]^j\right\}\\ &\leq 2C(\varepsilon),\end{aligned}$$
provided we take ε small such that $\gamma(\varepsilon) \leq 1/2$.

To prove the claim we construct $\{\phi_\ell\}$ by an induction. Note $\phi_0 = \{B_{1/2}\}$, independent of ε. Suppose $\phi_0, \phi_1, \ldots, \phi_{\ell-1}$ are already defined for some $\ell \geq 1$. To construct ϕ_ℓ, we take $B = B_r(y) \in \phi_{\ell-1}$, with $r \leq 1/2$. Consider the transformation $x \mapsto y + 2rx$. Then, via $\mathcal{L}u = 0$ in $B_{2r}(y)$, we have
$$\tilde{\mathcal{L}}\tilde{u} = 0 \quad \text{in} \quad B_1(0),$$
where
$$\tilde{\mathcal{L}} = \sum_{i,j=1}^{n} a_{ij}(y + 2rx)\partial_{x_ix_j} + \sum_{i=1}^{n} 2rb_i(y + 2rx)\partial_{x_i} + (2r)^2c(y + 2rx),$$
and
$$\tilde{u}(x) = u(y + 2rx).$$

Note Step 2 can be applied to $\tilde{\mathcal{L}}$ and $\tilde{u}$. Hence we obtain a collection of balls $\{B_{s_i}(z_i)\}$ with $s_i \le \varepsilon$ and $z_i \in \mathcal{S}(\tilde{u})$ such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(\tilde{u}) \cap B_{1/2} \setminus \cup B_{s_i}(z_i)\right) \le C(\varepsilon),$$

and

$$\sum s_i^{n-2} \le \gamma(\varepsilon).$$

Now transform $B_{1/2}$ back to $B_r(y)$ by $x \mapsto (x-y)/2r$. We obtain that for $B = B_r(y) \in \phi_{\ell-1}$, there exist finitely many balls $\{B_{r_i}(x_i)\}$ in $B_{2r}(y)$, with $r_i \le 2\varepsilon r$, such that

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap B_r(y) \setminus \bigcup_i B_{r_i}(x_i)\right) \le C(\varepsilon) r^{n-2},$$

and

$$\sum_i r_i^{n-2} \le r^{n-2}\gamma(\varepsilon).$$

Then we set

$$\phi_\ell^B = \bigcup_i \{B_i(x_i)\},$$

and

$$\phi_\ell = \bigcup_{B \in \phi_{\ell-1}} \phi_\ell^B.$$

Hence we obtain

$$\mathcal{H}^{n-2}\left(\mathcal{S}(u) \cap \bigcup_{B \in \phi_{\ell-1}} B \setminus \bigcup_{B \in \phi_\ell} B\right) \le C(\varepsilon)\left(\sum_{B_{r_i}(x_i) \in \phi_{\ell-1}} r_i^{n-2}\right),$$

and by an induction

$$r_i \le \frac{(2\varepsilon)^\ell}{2}, \qquad \sum_{B_{r_i}(x_i) \in \phi_\ell} r_i^{n-2} \le [\gamma(\varepsilon)]^\ell,$$

for each $\ell \ge 1$. This concludes the proof. □

3. Complex Singular Points of Planar Harmonic Functions

In the previous section, we derived a uniform estimate in terms of the frequency for the measure of singular sets to homogenous elliptic equation. Up to now, no explicit estimates are known even for harmonic functions. In this section, we shall derive an explicit estimate for planar harmonic functions.

Suppose u is a harmonic function defined in the unit ball in $\mathbb{R}^2$. Then u can be extended to a holomorphic function in some ball in $\mathbb{C}^2$. To see this, we simply consider the Taylor expansion of $u = u(x)$ at the origin and replace $x \in \mathbb{R}^2$ by $z \in \mathbb{C}^2$. With the estimate of the derivatives of harmonic functions, the new complex series converges for $|z| < R$, with $R \in (0,1)$ to be a universal constant. In the following, we always denote by $\tilde{u}$ the complexification of u. We shall also use $B_r(x)$ and $D_r(z)$ to denote open balls of radius r centered at x and z in $\mathbb{R}^2$ and $\mathbb{C}^2$, respectively. When the center is the origin, we will simply write B_r and D_r. The singular sets of u and $\tilde{u}$ are defined as

$$\begin{aligned}\mathcal{S}(u) =&\{x \in B_1; u(x) = \partial_{x_1}u(x) = \partial_{x_2}u(x) = 0\},\\ \mathcal{S}(\tilde{u}) =&\{z \in D_R; \tilde{u}(z) = \partial_{z_1}\tilde{u}(z) = \partial_{z_2}\tilde{u}(z) = 0\}.\end{aligned}$$

The main result in this section is the following theorem from [11].

Theorem 3.1. *Let u be a (real) harmonic function in $B_1 \subset \mathbb{R}^2$. Then for some universal constants $R_0 \in (0,1)$ and $c > 0$ there holds*

$$\#\left(\mathcal{S}(\tilde{u}) \cap D_{R_0}\right) \leq cN^2,$$

where N is defined as in (2.1).

A significant aspect of Theorem 3.1 is that a property of the complixified $\tilde{u}$ is determined by its restriction on the real space $u = \tilde{u}|_{\mathbb{R}^2}$. Here we make an important remark about the complexification $\tilde{u}$. Since u is a harmonic function, the holomorphic function $\tilde{u}$ satisfies

$$\partial_{z_1z_1}\tilde{u} + \partial_{z_2z_2}\tilde{u} = 0.$$

Theorem 3.1 asserts that the singular set of $\tilde{u}$ is isolated and that the number of singular points can be estimated in terms of the frequency of the (real) function u. This result does not hold for general holomorphic functions v satisfying

$$\partial_{z_1z_1}v + \partial_{z_2z_2}v = 0. \tag{3.1}$$

The following example is taken from [14].

Example 3.2. Let $v(z) = (z_1 - iz_2)^2$. Obviously v satisfies (3.1). However, the singular set of v is not even isolated.

Hence in order to have an isolated singular set for a holomorphic function $v = v(z_1, z_2)$ satisfying (3.1), all the coefficients in the Taylor expansion of v have to be real.

Now we begin to prove Theorem 3.1.

We first consider the gradient of homogeneous harmonic polynomials. We identify $\mathbb{R}^2 = \mathbb{C}$ and use the complex coordinate $z = x_1 + ix_2$. Consider the homogeneous polynomial

$$\bar{z}^d = (x_1 - ix_2)^d = r^d \cos d\theta - ir^d \sin d\theta.$$

We use its real part and complex part to construct a homogeneous polynomial map $Q_d : \mathbb{R}^2 \to \mathbb{R}^2$ as follows

$$Q_d(x) = Q_d(x_1, x_2) = \begin{pmatrix} r^d \cos d\theta \\ -r^d \sin d\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(z^d + \bar{z}^d) \\ \frac{i}{2}(z^d - \bar{z}^d) \end{pmatrix},$$

or

$$(3.2) \qquad Q_d(x) = \begin{pmatrix} \frac{1}{2}((x_1 + ix_2)^d + (x_1 - ix_2)^d) \\ \frac{i}{2}((x_1 + ix_2)^d - (x_1 - ix_2)^d) \end{pmatrix}.$$

Each component is a homogeneous harmonic polynomial. In fact Q_d is the gradient of some homogeneous harmonic polynomial of degree $d+1$. Now we extend the map $Q_d : \mathbb{C}^2 \to \mathbb{C}^2$ simply by replacing $x = (x_1, x_2)$ by $z = (z_1, z_2)$,

$$(3.3) \qquad Q_d(z) = Q_d(z_1, z_2) = \begin{pmatrix} \frac{1}{2}((z_1 + iz_2)^d + (z_1 - iz_2)^d) \\ \frac{i}{2}((z_1 + iz_2)^d - (z_1 - iz_2)^d) \end{pmatrix}.$$

We conclude easily

$$\begin{aligned} |Q_d(z)|^2 =& \frac{1}{2}\left(|z_1 + iz_2|^{2d} + |z_1 - iz_2|^{2d}\right) \\ =& \frac{1}{2}\Big(\left(|z_1|^2 + |z_2|^2 + 2(y_1x_2 - x_1y_2)\right)^d \\ &+ \left(|z_1|^2 + |z_2|^2 - 2(y_1x_2 - x_1y_2)\right)^d\Big). \end{aligned}$$

Notice that only the even power of $y_1x_2 - x_1y_2$ appears in the right side. Hence we get

$$(3.4) \qquad |Q_d(z)| \geq |z|^d.$$

Next we shall generalize (3.4) to nonhomogeneous harmonic polynomial maps.

Lemma 3.3. *Suppose P is a harmonic polynomial of degree $d+1$, with $P(0) = 0$ and $\int_{\mathbb{S}^1} P^2 \geq 1$. Then there exists an $r \in (1/2, 1)$ such that*

$$|\partial P(z)| > \varepsilon^d, \quad \textit{for any } z \in \partial D_r,$$

for some universal constant $\varepsilon \in (0,1)$.

The proof is based on a straightforward calculation. We omit the details.

Now, by Bezout formula, Lemma 3.3 and the 2-dimensional version of the Rouché Theorem, we obtain the following estimate.

Lemma 3.4. *Suppose that P is a harmonic polynomial of degree $d+1$, with $P(0) = 0$ and $\int_{\mathbb{S}^1} P^2 \geq 1$, and that $f : D_1 \subset \mathbb{C}^2 \to \mathbb{C}^2$ is holomorphic in D_1 and continuous up to the boundary ∂D_1. If for the universal $\varepsilon > 0$ in Lemma 3.3, there holds*

$$|f(z_1, z_2) - \partial P(z_1, z_2)| < \varepsilon^d, \quad \textit{for any } (z_1, z_2) \in D_1 \setminus D_{1/2},$$

then

$$\#\{f^{-1}(0) \cap D_{1/2}\} \leq d^2.$$

Next, we list some well known properties of harmonic functions. Suppose u is a harmonic function in $B_1 \subset \mathbb{R}^2$. For any $p \in B_1$, the frequency function $N(p, \cdot)$ at p is defined as

$$N(p,r) = \frac{r \int_{B_r(p)} |\nabla u|^2}{\int_{\partial B_r(p)} u^2}.$$

The frequency N in (2.1) is in fact $N(0,1)$.

The following result is exactly Theorem 1.1 in [16].

Theorem 3.5. *$N(p,r)$ is a monotone nondecreasing function of $r \in (0, 1 - |p|)$ for any $p \in B_1$.*

A corollary of this monotonicity is the doubling property, which we state only for $p = 0$. There holds for any $r \in (0, 1/2)$,

$$\frac{1}{2r}\int_{\partial B_{2r}} u^2 \leq 2^{2N(0,1)} \cdot \frac{1}{r}\int_{\partial B_r} u^2.$$

In fact, there holds a more general result for $0 < r_1 < r_2 \leq 1$

$$\frac{1}{r_2}\int_{\partial B_{r_2}} u^2 \leq \left(\frac{r_2}{r_1}\right)^{2N(0,1)} \cdot \frac{1}{r_1}\int_{\partial B_{r_1}} u^2. \tag{3.5}$$

For details, see [16].

We also need the following corollary of Theorem 3.5.

Corollary 3.6. *There exists a universal constant $N_0 \ll 1$ such that the following holds. If $N(0,1) \leq N_0$, then u does not vanish in $B_{1/2}$. If $N(0,1) \geq N_0$, then there holds*

$$N(p, \frac{1}{4}) \leq CN(0,1), \quad \textit{for any } p \in B_{\frac{1}{2}},$$

where C is a universal constant.

The proof follows exactly the same argument in the proof of Proposition 1.2 in [16] and is skipped. In fact, both assertions are proved there explicitly.

The second property we need is the complexification. Again, suppose u is a harmonic function in $B_1 \subset \mathbb{R}^2$. Then for some universal $R \in (0,1)$, u extends to a holomorphic function $\tilde{u}(z)$ in $D_R \subset \mathbb{C}^2$. Moreover, there holds for some universal constant $c > 0$

$$\sup_{D_R} |\tilde{u}| \leq c\|u\|_{L^2(\partial B_1)}. \tag{3.6}$$

In the following, R will be fixed such that the above extension property and (3.6) hold. Hence, the constant c is also fixed, independent of u.

Now we begin to prove Theorem 3.1. We shall prove the following result. The constant N in Theorem 3.7 means different from that in (2.1).

Theorem 3.7. *There are two universal constants $M > 1$ and $r \in (0,1)$ such that for a harmonic function u in $B_M \subset \mathbb{R}^2$, with $u(0) = 0$, satisfying*

$$\frac{M \int_{B_M} |\nabla u|^2}{\int_{\partial B_M} u^2} \le N,$$

there holds

$$\#\{z \in D_r; \tilde{u}_{z_1}(z) = \tilde{u}_{z_2}(z) = 0\} \le 4N^2.$$

Proof. For simplicity, we shall use the same notation to denote harmonic functions and their complexifications. Let (r,θ) denote polar coordinates in $\mathbb{R}^2$ and we write u in the following form

$$u(r,\theta) = \sum_{m=1}^{\infty} a_m \Phi_m(r,\theta), \quad \text{and} \quad \Phi_m(r,\theta) = r^m \varphi_m(\theta),$$

where $\varphi_m(\theta)$ satisfies

$$\int_{\mathbb{S}^1} \varphi_m^2(\theta) d\theta = 1, \quad \text{and} \quad \varphi_m''(\theta) + m^2 \varphi_m(\theta) = 0.$$

Moreover, we may assume, without loss of generality, that

$$\int_{\partial B_1} u^2 = \sum_{m=1}^{\infty} a_m^2 = 1. \tag{3.7}$$

In the following, we set

$$N_* = \inf\{n \in \mathbb{Z}_+;\ n \ge N\}.$$

In other words, $N_* = N$ if N is an integer and $N_* = [N] + 1$ otherwise. Here $[N]$ is the integral part of N. Obviously, we have

$$N_* - 1 \le N \le N_*.$$

By (3.5), we get

$$\frac{1}{M} \int_{\partial B_M} u^2 \le M^{2N(0,M)} \int_{\partial B_1} u^2 = M^{2N(0,M)},$$

which implies

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \le M^{2N(0,M)}.$$

By $N(0,M) \le N \le N_*$, we have obviously

$$\sum_{m=1}^{\infty} a_m^2 M^{2m} \le M^{2N_*}.$$

Therefore, we obtain

$$|a_m| \le M^{N_* - m}, \quad \text{for any } m \ge 1. \tag{3.8}$$

Since $\{\varphi_m\}$ is orthonormal in $L^2(\mathbb{S}^1)$, there holds for some universal constant $c > 0$

$$\int_{\partial B_1} |\sum_{m \geq 2N_*} a_m \Phi_m|^2 = \sum_{m \geq 2N_*} |a_m|^2 \leq \frac{c}{M^{2N_*}}.$$

We first choose M large, independent of N_*, such that

$$\sum_{m \geq 2N_*}^{\infty} |a_m|^2 \leq \frac{1}{2}. \tag{3.9}$$

By (3.6), we get for some universal $R \in (0,1)$,

$$\sup_{D_R} |\sum_{m \geq 2N_*} a_m \Phi_m| \leq \frac{c}{M^{N_*}}.$$

Interior estimates for holomorphic functions imply

$$\sup_{D_{R/2}} |\partial(\sum_{m \geq 2N_*} a_m \Phi_m)| \leq \frac{c}{RM^{N_*}}. \tag{3.10}$$

Set

$$P_* = \sum_{m=1}^{2N_*-1} a_m \Phi_m, \quad R_* = \sum_{m \geq 2N_*}^{\infty} a_m \Phi_m. \tag{3.11}$$

Then $u = P_* + R_*$. Obviously, we have by (3.7) and (3.9)

$$\sum_{m=1}^{2N_*-1} |a_m|^2 \geq \frac{1}{2}.$$

Then ∂P_* satisfies the assumptions in Lemma 3.3, with $d = 2N_* - 2$ and possibly a different normalization constant. By choosing M large enough, independent of N_*, we conclude by (3.10)

$$\sup_{D_{R/2}} |DR_*| < \varepsilon^{2N_*-2},$$

where ε is the universal constant as in Corollary 3.4, or Lemma 3.3. This implies

$$|\partial u(z) - \partial P_*(z)| < \varepsilon^{2N_*-2}, \quad \text{for any } z \in D_{R/2}.$$

By applying Corollary 3.4 to ∂u in $D_{R/2}$, we conclude that

$$\#\{|\partial u|^{-1}(0) \cap D_{R/4}\} \leq (2N_* - 2)^2.$$

This finishes the proof, since $N_* - 1 \leq N$. □

Now we may prove Theorem 3.1.

Proof of Theorem 3.1. Recall N is defined in (2.1).

First, we consider the case that N is small. Let N_0 be the constant in Corollary 3.6. If $N \leq N_0$, then u is never zero in $B_{1/4}$ by Corollary 3.6. Harnack inequality and interior estimates for harmonic functions and holomorphic functions imply that $\tilde{u}$ has no zeroes in D_{R_1}, for some universal $R_1 < 1$. Therefore we have $\mathcal{S}(\tilde{u}) \cap D_{R_1} = \phi$.

Next, we consider $N \geq N_0$. By Corollary 3.6, there holds for any $p \in B_{1/4}$

$$\frac{\int_{B_{\frac{1}{4}}(p)} |\nabla u|^2}{4 \int_{\partial B_{\frac{1}{4}}(p)} u^2} \leq CN,$$

for some positive constant C independent of u. For any $p \in B_{1/4}$, with $u(p) = 0$, by the scaled version of Theorem 3.7, we have

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{R_2}(p)\} \leq cN^2,$$

for some positive constants $R_2 < 1$ and c, independent of u and p. To finish the proof, we consider two cases. If u is never zero in $B_{R_2/2}$, then $\tilde{u}$ is never zero in $D_{2R_1R_2}$, as in the first part of the proof. This implies that $\mathcal{S}(\tilde{u}) \cap D_{2R_1R_2} = \phi$. If $u(p) = 0$ for some $p \in B_{R_2/2}$, then we have

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{R_2}(p)\} \leq cN^2,$$

which implies

$$\#\{\mathcal{S}(\tilde{u}) \cap D_{R_2/2}\} \leq cN^2.$$

This finishes the proof by taking $R_0 = \min\{R_1, 2R_1R_2, R_2/2\}$. □

To finish this section, we give an example to show that the number of complex singular points is indeed in the quadratic order of the frequency. Hence the estimate in Theorem 3.1 is optimal.

Example 3.8. For any integer $d \geq 2$ and any small $\varepsilon > 0$, consider the harmonic polynomial u in the polar coordinate

$$u(x) = \varepsilon r \cos\theta - \frac{1}{d+1} r^{d+1} \cos(d+1)\theta.$$

Then it is easy to see that

$$\partial u(x) = \begin{pmatrix} \varepsilon - r^d \cos d\theta \\ r^d \sin d\theta \end{pmatrix}.$$

By (3.3), we have

$$\partial \tilde{u}(z) = \begin{pmatrix} \varepsilon - \frac{1}{2}\left((z_1 + iz_2)^d + (z_1 - iz_2)^d\right) \\ -\frac{i}{2}\left((z_1 + iz_2)^d - (z_1 - iz_2)^d\right) \end{pmatrix}.$$

A simple calculation shows that $D\tilde{u}(z) = 0$ has d^2 solutions close to the origin. Obviously, the frequency of u is in the order of d.

References

[1] L. Bers, *Local behavior of solution of general linear elliptic equations,* Comm. Pure Appl. Math., **8**, 1955, 473-496.

[2] L.A. Caffarelli, and A. Friedman, *Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations,* J. Diff. Eq., **60**, 1985, 420-433.

[3] R.-T. Dong, *Nodal sets of eigenfunctions on Riemann surfaces,* J. Diff. Geom., **36**, 1992, 493-506.

[4] H. Donnelly, and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds,* Invent. Math., **93**, 1988, 161-183.

[5] H. Donnelly, and C. Fefferman, *Nodal sets for eigenfunctions of the Laplacian on surfaces,* J. Amer. Math. Soc., **3**, 1990, 333-353.

[6] H. Federer, *Geometric Measure Theory,* Springer-Verlag, New York, 1969.

[7] N. Garofalo, and F.-H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation,* Indiana Univ. Math J., **35**, 1986, 245-267.

[8] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Second Edition, Springer, Berlin, 1983.

[9] Q. Han, *Singular sets of solutions to elliptic equations,* Indiana Univ. Math J., **43**, 1994, 983-1002.

[10] Q. Han, *Schauder estimates for elliptic operators with applications to nodal sets,* J. Geom. Analysis, **10**, 2000, 455-480.

[11] Q. Han, *Singular sets of harmonic functions in $\mathbb{R}^2$ and their complexifications in $\mathbb{C}^2$,* to appear in Indiana Univ. Math J.

[12] Q. Han, R. Hardt, and F.-H. Lin, *Geometric measure of singular sets of elliptic equations* , Comm. Pure Appl. Math., **51**, 1998, 1425-1443.

[13] R. Hardt, M. Hoffman-Ostenhof, T. Hoffman-Ostenhof and N. Nadirashvili, *Critical sets of solutions to elliptic equations,* J. Diff. Geom., **51**, 1999, 359-373.

[14] M. Hoffman-Ostenhof, T. Hoffman-Ostenhof and N. Nadirashvili, *Critical sets of smooth solutions to elliptic equations in dimension 3,* Indiana Univ. Math J., **45**, 1996, 15-37.

[15] B. Levin, *Distribution of Zeros of Entire Functions,* Translations of Mathematical Monographs, Vol.5, AMS, 1964.

[16] F.-H. Lin, *Nodal sets of solutions of elliptic and parabolic equations,* Comm. Pure Appl. Math., **44**, 1991, 287-308.

[17] G. Lupacciolu, *A Rouché type theorem in several complex variables,* Rend. Accad. Naz. Sci. XL Mem. Mat., **5**, 1985, 33-41.

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